TITLE:

水面孤立波の斜め相互作用の理論解 : 片方の振幅が大きい場合 (非線形波動現象の研究の新たな進展)

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水面孤立波の斜め相互作用の理論解
—片方の振幅が大きい場合—

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要旨

２つの水面孤立波が斜め相互作用するときの様子を Euler 方程式系を基に調べ、片方の波の振幅が大きくもう片方の波の振幅が小さい場合の理論解を求めた。理論解の導出過程を示した後に、衝突による位相のずれや、放出波などについて、得られた理論解を基にした考察を行う。

1. 緒言

２つの水面孤立波が斜め相互作用するときの様子を Euler 方程式系を基に調べ、片方の波の振幅が大きくもう片方の波の振幅が小さい場合の理論解を求める。まずは２節で基礎方程式と孤立波解を提示する. 続く３節で共鳴しない場合の理論解の導出過程を示しており、最後の合わせて考察も行っている。なお、講演時に紹介した共鳴する場合の理論解については、今回は紙面の都合上割愛する。また, 関連する過去の研究として Johnson [1] が同様の条件下で孤立波の斜め相互作用の解を導いている. 彼の理論解と本理論解との具体的な関連については, 現在, 調査中である.

2. 基礎方程式と孤立波解

We consider three-dimensional irrotational motion of an incompressible ideal fluid with a free surface under the uniform acceleration g due to gravity. The fluid lies on a flat bottom and has undisturbed depth D. The effects of surface tension are neglected. In what follows, all variables are non-dimensionalized using g and D. Introducing the three-dimensional Cartesian coordinates x, y, z with z vertically upward and their origin located at the bottom, we obtain the following set of dimensionless governing equations for the fluid motion:

$$\nabla^2\phi = 0 \quad \text{for} \quad 0 < z < \eta, \tag{2.1}$$

with boundary conditions

$$\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y} - \frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z = \eta, \tag{2.2}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + \eta = b(t) \quad \text{at} \quad z = \eta, \tag{2.3}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z = 0, \tag{2.4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{2.5}$$

t is the time, $\phi(x,y,z,t)$ is the velocity potential, and $\eta(x,y,t)$ is the surface elevation and $b(t)$ is a function of t which is determined by evaluating (2.3) as $x \to \infty$.

Consider a steadily propagating solution of (2.1)-(2.4) that is independent of t and y:

$$\phi = -vx + \Phi(x,z;v), \quad \eta = \mathrm{H}(x;v), \tag{2.6a,b}$$

where v is a positive parameter, and

$$\frac{\partial \Phi}{\partial x} \to 0, \ \frac{\partial \Phi}{\partial z} \to 0, \ \mathrm{H} - 1 \to 0 \quad \text{as } x \to \pm\infty. \tag{2.6c}$$

Solution (2.6) represents a steady propagation of a two-dimensional localized wave against a uniform stream of constant velocity $-v$ in the x direction. We call this solution a solitary wave solution. From substitution of (2.6) into (2.1)-(2.4), Φ and H should satisfy

$$\nabla_\perp^2 \Phi = 0 \quad \text{for } 0 < z < \mathrm{H}, \tag{2.7}$$

$$\left(-v + \frac{\partial \Phi}{\partial x}\right)\frac{\mathrm{dH}}{\mathrm{d}x} = \frac{\partial \Phi}{\partial z} \quad \text{at } z = \mathrm{H}, \tag{2.8}$$

$$-v\frac{\partial \Phi}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right] + \mathrm{H} = 0 \quad \text{at } z = \mathrm{H}, \tag{2.9}$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{at } z = 0, \tag{2.10}$$

in addition to (2.6c), where

$$\nabla_\perp^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{2.10b}$$

The existence of the above solitary wave solution was rigorously proved by Amick & Toland [2]. Numerical solutions were obtained by Hunter & Vanden-Broeck [3] and many others (Byatt-Smith & Longuet-Higgins [4]; Tanaka [5]; Longuet-Higgins & Tanaka [6]). According to them, solitary wave solution exists in the range $1 < v \le 1.2942$, or $0 < a \le 0.83332$, where a is the maximum surface elevation which is called amplitude hereafter. The solitary wave solution has the property that the surface elevation $\mathrm{H} - 1$ is positive and possesses a single point of extremum which is called the crest. Moreover the solution is symmetric with respect to the crest, that is, $\Phi(x,z)$ is odd and $\mathrm{H}(x)$ is even in x with $x = 0$ on the crest, and approach to the state at infinity as $x \to \pm\infty$ described by (2.6c) is exponentially fast (some of these properties were rigorously proved by Amick & Toland [2] and Craig & Sternberg [7]). In fact, Φ approaches different constant values as $x \to \pm\infty$, i.e.

$$C(v) \equiv [\Phi]_{x\to\infty} - [\Phi]_{x\to-\infty} = vM - \frac{2T}{v}, \tag{2.11a}$$

where M and T are defined by

$$M(v) = \int_{-\infty}^{\infty}(\mathrm{H}-1)\mathrm{d}x, \quad T(v) = \int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\mathrm{H}}\frac{1}{2}\left[\left(\frac{\partial \Phi_s}{\partial x}\right)^2 + \left(\frac{\partial \Phi_s}{\partial z}\right)^2\right]\mathrm{d}z, \tag{2.11b,c}$$

and the far right side of (2.11a) was first derived by McCowan [8]. C, M, and T physically represent the circulation, the mass, and the kinetic energy of the solitary wave, respectively. Incidentally, the total energy E of the solitary wave is defined as

$$E(v)=T(v)+\frac{1}{2}\int_{-\infty}^{\infty}(\mathrm{H}-1)^2\mathrm{d}x=\int_{-\infty}^{\infty}\mathrm{d}x\int_0^H\left(\frac{\partial\Phi}{\partial x}\right)^2\mathrm{d}z\,, \tag{2.11d}$$

where the far right side was first derived by Starr [9] (see also Appendix A in Kataoka [10]).

When amplitude is small, or $a=\varepsilon^2$ with ε being a small parameter, Φ and $\mathrm{H}-1$ are small quantities of $O(\varepsilon)$ and $O(\varepsilon^2)$, respectively, and the dependency on x is slow (appreciable variation in x of the order of ε^{-1}). Let us call this solution a small-amplitude solitary wave solution and denote it by $\left(\varepsilon\Phi_s(X,z),1+\varepsilon^2\mathrm{H}_s(X)\right)$, where

$$X=\varepsilon x \tag{2.12}$$

is a shrunk coordinate in x. The solution for $a=\varepsilon^2$ is then given by (Grimshaw [11] with some transcript errors, Fenton [12])

$$\varepsilon\Phi_s(X,z)=\varepsilon\Phi_{s1}(X)+\varepsilon^3\Phi_{s3}(X,z)+\cdots, \tag{2.13a}$$

$$1+\varepsilon^2\mathrm{H}_s(X)=1+\varepsilon^2\mathrm{H}_{s2}(X)+\varepsilon^4\mathrm{H}_{s4}(X)+\cdots, \tag{2.13b}$$

$$v=1+\frac{\varepsilon^2}{2}-\frac{3}{20}\varepsilon^4+\frac{3}{56}\varepsilon^6+\cdots, \tag{2.13c}$$

where

$$\Phi_{s1}(X)=\frac{2\sqrt{3}}{3}\tanh(kX),\quad \Phi_{s3}(X,z)=\sqrt{3}\left[\frac{5}{36}+\left(-\frac{2}{9}+\frac{z^2}{2}\right)\mathrm{sech}^2(kX)\right]\tanh(kX), \tag{2.14a,b}$$

$$\Phi_{s5}(X,z)=\sqrt{3}\left[-\frac{419}{1600}+\left(\frac{3}{100}-\frac{3}{16}z^2-\frac{z^4}{8}\right)\mathrm{sech}^2(kX)+\left(\frac{4}{25}-z^2+\frac{3}{8}z^4\right)\mathrm{sech}^4(kX)\right]\tanh(kX), \tag{2.14c}$$

$$\mathrm{H}_{s2}(X)=\mathrm{sech}^2(kX),\quad \mathrm{H}_{s4}(X)=-\frac{3}{4}\mathrm{sech}^2(kX)\tanh^2(kX), \tag{2.14d,e}$$

$$\mathrm{H}_{s6}(X)=\left[\frac{5}{8}\mathrm{sech}^2(kX)-\frac{101}{80}\mathrm{sech}^4(kX)\right]\tanh^2(kX), \tag{2.14f}$$

and

$$k=\frac{\sqrt{3}}{2}\left(1-\frac{5}{8}\varepsilon^2+\frac{71}{128}\varepsilon^4+\cdots\right). \tag{2.14g}$$

3. 孤立波の斜め相互作用（非共鳴）の理論解導出

Consider interactions between two obliquely moving solitary waves, one of which has finite amplitude a (wave speed v) and the other has small amplitude ε^2 (wave speed $1+\varepsilon^2/2+\cdots$). We take a reference frame moving with the undisturbed finite-amplitude solitary wave whose traveling direction is in the positive x direction and crest is on $x=0$. Small-amplitude solitary wave propagates at an inclination angle ψ to the x axis ($0\le\psi\le\pi$)（図 1 参照）. The solution before interaction is a summation of the two solitary wave solutions:

$$\begin{cases}\phi=-vx+\Phi(x,z;v)+\varepsilon\Phi_s\left(\theta+O(\varepsilon^2),z\right)\\ \eta=\mathrm{H}(x;v)+\varepsilon^2\mathrm{H}_s\left(\theta+O(\varepsilon^2)\right)\end{cases}\quad\text{(before interaction)}, \tag{3.1}$$

where

$$\theta=X\cos\psi+(Y-c_y\tau)\sin\psi\,, \tag{3.2}$$

and

$$X = \varepsilon x,\ \ Y = \varepsilon y,\ \ \tau = \varepsilon t \tag{3.3}$$

are shrunk coordinates in x, y and t, respectively. c_y is the leading-order wave speed in the y direction (or along the crest of the finite-amplitude solitary wave) of the small-amplitude solitary wave given by（図 1 参照）

$$c_y = \frac{1 - v\cos\psi}{\sin\psi}. \begin{pmatrix} < 0 \ \text{ for } 0 \le \psi < \cos^{-1}(1/v) \\ > 0 \ \text{ for } \cos^{-1}(1/v) < \psi \le \pi \end{pmatrix} \tag{3.4}$$

Since any small perturbations propagate in the negative X direction in this reference frame, the solution (3.1) before interaction becomes the boundary condition as $X \to \infty$, i.e.

$$\begin{cases} \phi = -vx + [\Phi]_{x \to \pm\infty} + \varepsilon \Phi_s\left(\theta + O(\varepsilon^2), z\right) \\ \eta = 1 + \varepsilon^2 \mathrm{H}_s\left(\theta + O(\varepsilon^2)\right) \end{cases} \text{ as } X \to \infty. \tag{3.5}$$

Here we look for an asymptotic solution for small ε of (2.1)-(2.4) whose boundary condition as $X \to \infty$ is (3.5).

Scale of variation in y and t of the interaction process is $O(\varepsilon^{-1})$ from the functional form of (3.1) which depends on y and t only through $Y - c_y\tau$. For the scale of variation in x, two different scales coexist: $O(1)$ due to the finite-amplitude solitary wave solution and $O(\varepsilon^{-1})$ due to the small-amplitude one. We therefore seek solutions of different scales of variation in x: a solution with an appreciable variation in x of $O(1)$ (core solution) and that with an appreciable variation in x of $O(\varepsilon^{-1})$ (far-field solution). Scale of variation in y and t (or $y - c_y t$) is fixed at $O(\varepsilon^{-1})$. The above two solutions are looked for in Sections 3.1 and 3.2, respectively, and they are unified to an overall solution by matching procedure in Section 3.3.

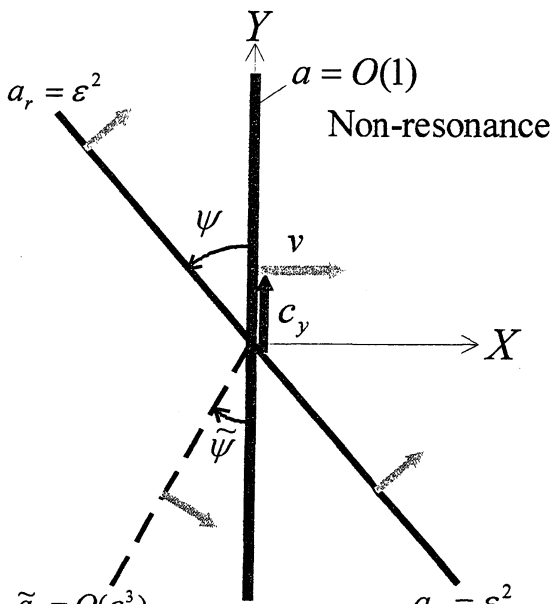

図 1 有限振幅孤立波（振幅 $a = O(1)$ ；峰が太い実線）と小振幅孤立波（振幅 $a_S = \varepsilon^2 << 1$ ；峰が細い実線）の斜め相互作用の模式図.

3.1. *Core solution*

Following the statement in the previous paragraph, we look for a solution of (2.1)-(2.4) with a moderate variation in x, $Y-c_y\tau$ and z in the following power series of ε:

$$\phi_C = -vx + \Phi(x,z;v) + \varepsilon\phi_{C1}(x,Y-c_y\tau,z) + \varepsilon^2\phi_{C2}(x,Y-c_y\tau,z) + \cdots, \tag{3.6a}$$

$$\eta_C = \mathrm{H}(x;v) + \varepsilon\eta_{C1}(x,Y-c_y\tau) + \varepsilon^2\eta_{C2}(x,Y-c_y\tau) + \cdots, \tag{3.6b}$$

where each component function is of the order of unity ($\phi_{Cn} = O(1)$, $\eta_{Cn} = O(1)$) and the subscript C is attached to (ϕ,η) in order to indicate the type of solution (core solution). Substituting the solution (3.6a,b) into (2.1)-(2.4) and arranging the same-order terms in ε, we obtain a series of sets of equations for (ϕ_{Cn},η_{Cn}) ($n=1,2,\cdots$) as

$$\nabla_\perp^2\phi_{Cn} = F_n \equiv \begin{cases} 0 & (n=1,2) \\ -\dfrac{\partial^2\phi_{Cn-2}}{\partial Y^2} & (n\ge 3) \end{cases} \quad \text{for } 0<z<\mathrm{H}, \tag{3.7}$$

$$\boldsymbol{L}_K[\phi_{Cn},\eta_{Cn}] = G_n \equiv \begin{cases} 0 & (n=1) \\ -\dfrac{\partial\eta_{Cn-1}}{\partial\tau} & (n=2,3) \end{cases} \quad \text{at } z=\mathrm{H}, \tag{3.8}$$

$$\boldsymbol{L}_D[\phi_{Cn},\eta_{Cn}] = I_n \equiv \begin{cases} 0 & (n=1) \\ -\dfrac{\partial\phi_{Cn-1}}{\partial\tau} & (n=2,3) \end{cases} \quad \text{at } z=\mathrm{H}, \tag{3.9}$$

$$\frac{\partial\phi_{Cn}}{\partial z} = 0 \quad \text{at } z=0, \tag{3.10}$$

where $\boldsymbol{L}_K$ and $\boldsymbol{L}_D$ are the linear operators defined by

$$\boldsymbol{L}_K[\phi,\eta] = \left(-\frac{\partial}{\partial z} + \frac{\mathrm{dH}}{\mathrm{d}x}\frac{\partial}{\partial x}\right)\phi + \left(\frac{\partial}{\partial x} + \frac{\mathrm{dH}}{\mathrm{d}x}\frac{\partial}{\partial z}\right)\left[\left(-v + \frac{\partial\Phi}{\partial x}\right)\eta\right], \tag{3.11a}$$

$$\boldsymbol{L}_D[\phi,\eta] = \left[\left(-v + \frac{\partial\Phi}{\partial x}\right)\frac{\partial}{\partial x} + \frac{\partial\Phi}{\partial z}\frac{\partial}{\partial z}\right]\phi + \left[\left(-v + \frac{\partial\Phi}{\partial x}\right)\frac{\partial^2\Phi}{\partial x\partial z} + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z^2} + 1\right]\eta. \tag{3.11b}$$

For $n=1$, the above set of equations (3.7)-(3.10) is homogeneous, and has a solution of the following form:

$$\phi_{C1} = -\frac{\partial\Phi}{\partial x}p(Y-c_y\tau) - q(Y-c_y\tau), \quad \eta_{C1} = -\frac{\mathrm{dH}}{\mathrm{d}x}p(Y-c_y\tau), \tag{3.12a,b}$$

where p and q are undetermined functions of $Y-c_y\tau$. The first terms on the right-hand sides of (3.10) come from invariance of the system (2.7)-(2.10) under the horizontal shift.

For $n=2,3,\cdots$, the set of equations (3.7)-(3.10) is linear and inhomogeneous. For this set of linear inhomogeneous equations to have a solution, its inhomogeneous terms on the right-hand sides of (3.2)-(3.4) must satisfy some relation. Since the homogeneous part satisfies

$$\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\mathrm{H}}\frac{\partial\Phi}{\partial x}\nabla_\perp^2\phi_{Cn}\mathrm{d}z + \int_{-\infty}^{\infty}\left[\frac{\partial\Phi}{\partial x}\boldsymbol{L}_K[\phi_{Cn},\eta_{Cn}] - \frac{\mathrm{dH}}{\mathrm{d}x}\boldsymbol{L}_D[\phi_{Cn},\eta_{Cn}]\right]_{z=\mathrm{H}}\mathrm{d}x = 0,$$

its inhomogeneous terms F_n, G_n and I_n on the right-hand sides of (3.7)-(3.9) must satisfy the following relation (solvability condition):

$$\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\mathrm{H}}\frac{\partial\Phi}{\partial x}F_n\mathrm{d}z + \int_{-\infty}^{\infty}\left[\frac{\partial\Phi}{\partial x}G_n - \frac{\mathrm{dH}}{\mathrm{d}x}I_n\right]_{z=\mathrm{H}}\mathrm{d}x = 0, \tag{3.13}$$

where the quantities in the square brackets with subscript $z=\mathrm{H}$ (or $[\ ]_{z=\mathrm{H}}$) are evaluated at $z=\mathrm{H}$.

For $n=2$, the solvability condition (3.13) is identically satisfied, and the solution of (3.7)-(3.10) for $n=2$ is

$$\phi_{C2}=\frac{\partial\Phi}{\partial v}\frac{\partial p}{\partial\tau}+\Phi_B\frac{\partial q}{\partial\tau}+\Phi_U r(Y-c_y\tau),\quad \eta_{C2}=\frac{\partial\mathrm{H}}{\partial v}\frac{\partial p}{\partial\tau}+\mathrm{H}_B\frac{\partial q}{\partial\tau}+\mathrm{H}_U r(Y-c_y\tau),\quad (3.14a,b)$$

where r is a new undetermined function of $Y-c_y\tau$, and $\partial(\Phi,\mathrm{H})/\partial v$ represents the derivative of (Φ,H) with respect to v keeping x and z constant. (Φ_B,H_B) is a particular solution of (3.7)-(3.10) with $F_n=G_n=0$ and $I_n=1$ which remains finite as $x\to\pm\infty$, while (Φ_U,H_U) is that of (3.7)-(3.10) with $F_n=G_n=I_n=0$ which diverges like x as $x\to\pm\infty$. Specifically, (Φ_B,H_B) and (Φ_U,H_U) are given by

$$\Phi_B=\frac{3}{2}\Phi-x\frac{\partial\Phi}{\partial x}-z\frac{\partial\Phi}{\partial z}-\frac{v}{2}\frac{\partial\Phi}{\partial v},\quad \mathrm{H}_B=\mathrm{H}-x\frac{\mathrm{dH}}{\mathrm{d}x}-\frac{v}{2}\frac{\partial\mathrm{H}}{\partial v},\qquad (3.15a,b)$$

$$\Phi_U=x+v\left(\frac{3}{2}\Phi-x\frac{\partial\Phi}{\partial x}-z\frac{\partial\Phi}{\partial z}\right)-\left(\frac{v^2}{2}+1\right)\frac{\partial\Phi}{\partial v}=x-\frac{\partial\Phi}{\partial v}+v\Phi_B,\qquad (3.15c)$$

$$\mathrm{H}_U=v\left(\mathrm{H}-x\frac{\mathrm{dH}}{\mathrm{d}x}\right)-\left(\frac{v^2}{2}+1\right)\frac{\partial\mathrm{H}}{\partial v}=-\frac{\partial\mathrm{H}}{\partial v}+v\mathrm{H}_B.\qquad (3.15d)$$

The solution ϕ_{C2} in (3.14a) satisfies

$$[\phi_{C2}-rx]_{x\to\infty}-[\phi_{C2}-rx]_{x\to-\infty}=\frac{\mathrm{d}C}{\mathrm{d}v}\frac{\partial p}{\partial\tau}+C_B\frac{\partial q}{\partial\tau}+C_U r,\qquad (3.16)$$

where C, C_B and C_U are defined by (2.11a) and

$$C_B=\frac{3}{2}C-\frac{v}{2}\frac{\mathrm{d}C}{\mathrm{d}v},\quad C_U=-\frac{\mathrm{d}C}{\mathrm{d}v}+vC_B.\qquad (3.17a,b)$$

For $n=3$ the solvability condition (3.13) becomes

$$\left(\frac{1}{v}\frac{\mathrm{d}E}{\mathrm{d}v}\frac{\partial^2}{\partial\tau^2}-E\frac{\partial^2}{\partial Y^2}\right)p=\left[\left(v\frac{\mathrm{d}C}{\mathrm{d}v}-\frac{\mathrm{d}M}{\mathrm{d}v}\right)\frac{\partial^2}{\partial\tau^2}+vM\frac{\partial^2}{\partial Y^2}\right]q+(v^2-1)\frac{\mathrm{d}C}{\mathrm{d}v}\frac{\partial r}{\partial\tau},\qquad (3.18)$$

where E, M and C are defined by (2.11), and the relation

$$\int_{-\infty}^{\infty}\left[\frac{\partial\Phi}{\partial x}\eta_{C2}-\frac{\mathrm{dH}}{\mathrm{d}x}\phi_{C2}\right]_{z=\mathrm{H}}\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}C}{\mathrm{d}v}\left([u_{C2}]_{x\to-\infty}+[u_{C2}]_{x\to\infty}\right)$$
$$+\int_{-\infty}^{\infty}\left\{-\int_0^{\mathrm{H}}\frac{\partial\Phi}{\partial v}\nabla_\perp^2\phi_{C2}\mathrm{d}z+\left[-\frac{\partial\Phi}{\partial v}\mathrm{L}_{\mathrm{K}0}[\phi_{C2},\eta_{C2}]+\frac{\partial\mathrm{H}}{\partial v}\mathrm{L}_{\mathrm{D}0}[\phi_{C2},\eta_{C2}]\right]_{z=\mathrm{H}}\right\}\mathrm{d}x\qquad (3.19)$$

with

$$u_{Cn}=\int_0^{\mathrm{H}}\frac{\partial\phi_{Cn}}{\partial x}\mathrm{d}z-v\eta_{Cn},\qquad (3.20)$$

is used to derive (3.18). When the condition (3.18) is satisfied, a solution for $n=3$ exists, and from integration of (3.7) with respect to x and z and use of (3.8), one finds

$$\left[u_{C3}-\frac{\partial u_{C3}}{\partial x}x\right]_{x\to\infty}-\left[u_{C3}-\frac{\partial u_{C3}}{\partial x}x\right]_{x\to-\infty}=\left(-\frac{\mathrm{d}M}{\mathrm{d}v}\frac{\partial^2}{\partial\tau^2}+vM\frac{\partial^2}{\partial Y^2}\right)p-\left(M_B\frac{\partial^2}{\partial\tau^2}-M\frac{\partial^2}{\partial Y^2}\right)q-M_U\frac{\partial r}{\partial\tau},$$
$$(3.21)$$

where M, M_B and M_U are defined by (2.11b) and

$$M_B = 2M - \frac{v}{2}\frac{\mathrm{d}M}{\mathrm{d}v}, \quad M_U = -\frac{\mathrm{d}M}{\mathrm{d}v} + vM_B, \tag{3.22a,b}$$

3.2. *Far-field solution*

We look for a solution of (2.1)-(2.4) with a moderate variation in $X(=\varepsilon x)$, $Y - c_y\tau$ and z in the following power series of ε:

$$\phi_F = -vx + [\Phi]_{x\to\pm\infty} + \varepsilon\phi_{F1}(X, Y - c_y\tau, z) + \varepsilon^2\phi_{F2}(X, Y - c_y\tau, z) + \cdots, \tag{3.23a}$$

$$\eta_F = 1 + \varepsilon^2\eta_{F2}(X, Y - c_y\tau) + \varepsilon^3\eta_{F3}(X, Y - c_y\tau) + \cdots, \tag{3.23b}$$

where each component function is of the order of unity ($\phi_{Fn} = O(1)$, $\eta_{Fn} = O(1)$), and the subscript F is attached to (ϕ, η) in order to indicate the type of solution (far-field solution). The series of (3.23) start from $O(\varepsilon)$ and $O(\varepsilon^2)$ for $\phi_F + vx - [\Phi]_{x\to\pm\infty}$ and $\eta_F - 1$, respectively, in accordance with the core solution having nonzero values as $x \to \pm\infty$ from these orders (see (3.12a) and (3.14b)). Substituting (3.23) into (2.1)-(2.4) and arranging the same-order terms in ε, we obtain a series of equations for ϕ_{Fn} ($n = 1, 2, \cdots$). For $n = 1$ and 2, they are homogeneous ($\partial^2\phi_{Fn}/\partial z^2 = 0$ for $0 < z < 1$ and $\partial\phi_{Fn}/\partial z = 0$ at $z = 0$ and 1) and have a solution independent of z:

$$\phi_{Fn} = \phi_{Fn}(X, Y - c_y\tau) \quad (n = 1 \text{ and } 2). \tag{3.24}$$

For $n = 3$ and 4, the equations are inhomogeneous, i.e.

$$\frac{\partial^2\phi_{Fn}}{\partial z^2} = J_n \equiv -\frac{\partial^2\phi_{Fn-2}}{\partial X^2} - \frac{\partial^2\phi_{Fn-2}}{\partial Y^2} \quad \text{for } 0 < z < 1, \tag{3.25}$$

$$\frac{\partial\phi_{Fn}}{\partial z} = K_n \equiv \left(\frac{\partial}{\partial\tau} - v\frac{\partial}{\partial X}\right)\eta_{Fn-1} \quad \text{at } z = 1, \tag{3.26}$$

$$\frac{\partial\phi_{Fn}}{\partial z} = 0 \quad \text{at } z = 0, \tag{3.27}$$

where

$$\eta_{Fn-1} = -\left(\frac{\partial}{\partial\tau} - v\frac{\partial}{\partial X}\right)\phi_{Fn-2}. \tag{3.28}$$

For the above inhomogeneous equations (3.25)-(3.27) to have a solution, their inhomogeneous terms J_n and K_n on the right-hand sides of (3.25) and (3.26) must satisfy the solvability condition:

$$\int_0^1 J_n \mathrm{d}z = K_n, \tag{3.29}$$

which gives

$$\left[\left(\frac{\partial}{\partial\tau} - v\frac{\partial}{\partial X}\right)^2 - \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)\right]\phi_{Fn-2} = 0 \quad (n = 3 \text{ and } 4). \tag{3.30}$$

With the aid of (3.4) and the boundary condition (3.5) as $X \to \infty$, (3.30) leads to

$$\phi_{F1} = \begin{cases} \Phi_{s1}(\theta) & \text{for } X > 0, \\ \varphi_1(\theta) + \tilde{\varphi}_1(\tilde{\theta}) & \text{for } X < 0, \end{cases} \tag{3.31}$$

$$\phi_{F2} = \begin{cases} 0 & \text{for } X > 0, \\ \varphi_2(\theta) + \widetilde{\varphi}_2(\widetilde{\theta}) & \text{for } X < 0, \end{cases} \tag{3.32}$$

where Φ_{s1} is given by (2.14a). Here φ_1, $\widetilde{\varphi}_1$, φ_2 and $\widetilde{\varphi}_2$ are undetermined functions of θ or $\widetilde{\theta}$ with

$$\theta = X\cos\psi + (Y - c_y\tau)\sin\psi, \quad \widetilde{\theta} = X\cos\widetilde{\psi} + (Y - c_y\tau)\sin\widetilde{\psi}, \tag{3.33}$$

and ψ $(0 \le \psi \le \pi)$ and $\widetilde{\psi}$ $(-\pi \le \widetilde{\psi} \le 0)$ are two solutions of (3.4) for ψ (図 1 参照), i.e.

$$\psi = \cos^{-1}\frac{v - c_y\sqrt{v^2 + c_y^2 - 1}}{v^2 + c_y^2} = \sin^{-1}\frac{c_y + v\sqrt{v^2 + c_y^2 - 1}}{v^2 + c_y^2}(\ge 0), \tag{3.34a}$$

$$\widetilde{\psi} = \cos^{-1}\frac{v + c_y\sqrt{v^2 + c_y^2 - 1}}{v^2 + c_y^2} = \sin^{-1}\frac{c_y - v\sqrt{v^2 + c_y^2 - 1}}{v^2 + c_y^2}(\le 0). \tag{3.34b}$$

η_{F2} and η_{F3} are obtained from substitution of (3.31) and (3.32) into (3.28) as

$$\eta_{F2} = \begin{cases} \dfrac{\mathrm{d}\Phi_{s1}(\theta)}{\mathrm{d}\theta} & \text{for } X > 0, \\ \dfrac{\mathrm{d}\varphi_1}{\mathrm{d}\theta} + \dfrac{\mathrm{d}\widetilde{\varphi}_1}{\mathrm{d}\widetilde{\theta}} & \text{for } X < 0, \end{cases} \qquad \eta_{F3} = \begin{cases} 0 & \text{for } X > 0, \\ \dfrac{\mathrm{d}\varphi_2}{\mathrm{d}\theta} + \dfrac{\mathrm{d}\widetilde{\varphi}_2}{\mathrm{d}\widetilde{\theta}} & \text{for } X < 0. \end{cases} \tag{3.35a,b}$$

3.3 *Matching*

We carry out matching of the core solution (ϕ_C, η_C) and the far-field solution (ϕ_F, η_F). In the core region expressed by (ϕ_C, η_C), the ordering of the far-field solution is rearranged. Specifically, the far-field solution (ϕ_F, η_F) is expanded in the power series of X (or εx):

$$h_{Fn} = (h_{Fn})_0 + \varepsilon x\left(\frac{\partial h_{Fn}}{\partial X}\right)_0 + \frac{\varepsilon^2 x^2}{2}\left(\frac{\partial^2 h_{Fn}}{\partial X^2}\right)_0 + \cdots, \tag{3.36}$$

where h represents (ϕ, η), and the quantities in the parentheses with subscript 0 are evaluated at $X = 0$. We then collect the same orders of ε and obtain the reordered form [say, $(\phi^*_{Fn}, \eta^*_{Fn})$] of (ϕ_{Fn}, η_{Fn}). Matching is carried out by comparing the forms of the two solutions (ϕ_{Cn}, η_{Cn}) and $(\phi^*_{Fn}, \eta^*_{Fn})$ at each n from $n = 1$, and it is accomplished if the conditions

$$[\phi_{Cn}]_{x\to\pm\infty} = \phi^*_{Fn}, \quad [\eta_{Cn}]_{x\to\pm\infty} = \eta^*_{Fn},$$

are satisfied.

For $n = 1$, since $\phi^*_{F1} = (\phi_{F1})_0$, the matching conditions are, from (3.12a) and (3.31),

$$-q = \Phi_{s1}(\theta_0), \tag{3.37a}$$

$$-q = \varphi_1(\theta_0) + \widetilde{\varphi}_1(\widetilde{\theta}_0), \tag{3.37b}$$

where

$$\theta_0 = (Y - c_y\tau)\sin\psi, \quad \widetilde{\theta}_0 = (Y - c_y\tau)\sin\widetilde{\psi}. \tag{3.37c}$$

For $n = 2$, since $\phi^*_{F2} = (\phi_{F2})_0 + x(\partial\phi_{F1}/\partial X)_0$, the matching conditions are composed of two different kinds of terms, i.e. those independent of x and those proportional to x. From those independent of x, we have

$$[\phi_{C2} - rx]_{x\to\infty} = 0, \tag{3.39a}$$

$$[\phi_{C2} - rx]_{x\to-\infty} = \varphi_2(\theta_0) + \tilde{\varphi}_2(\tilde{\theta}_0), \tag{3.39b}$$

where (3.14a) and (3.32) are used. From those proportional to x,

$$r = \frac{\mathrm{d}\Phi_{s1}(\theta_0)}{\mathrm{d}\theta_0}\cos\psi, \tag{3.40a}$$

$$r = \frac{\mathrm{d}\varphi_1(\theta_0)}{\mathrm{d}\theta_0}\cos\psi + \frac{\mathrm{d}\tilde{\varphi}_1(\tilde{\theta}_0)}{\mathrm{d}\tilde{\theta}_0}\cos\tilde{\psi}, \tag{3.40b}$$

where (3.14a) and (3.31) are used.

For $n = 3$, where $\phi^{*}_{F3} = (\phi_{F3})_0 + x(\partial\phi_{F2}/\partial X)_0 + x^2(\partial^2\phi_{F1}/\partial X^2)_0/2$, the matching conditions proportional to x contribute to determination of unknowns at this stage. It is convenient to express them in terms of u_{C3} defined by (3.20), i.e.

$$\left[u_{C3} - \frac{\partial u_{C3}}{\partial x}x\right]_{x\to\infty} = 0, \tag{3.41a}$$

$$\left[u_{C3} - \frac{\partial u_{C3}}{\partial x}x\right]_{x\to-\infty} = (\cos\psi - v)\frac{\mathrm{d}\varphi_2(\theta_0)}{\mathrm{d}\theta_0} + (\cos\tilde{\psi} - v)\frac{\mathrm{d}\tilde{\varphi}_2(\tilde{\theta}_0)}{\mathrm{d}\tilde{\theta}_0}, \tag{3.41b}$$

where (3.32) and (3.36) are used. Matching conditions for η are automatically satisfied if (3.40) and (3.41) are satisfied.

Thus, the four unknowns q, r, $\varphi_1(\theta_0)$ and $\tilde{\varphi}_1(\tilde{\theta}_0)$ are determined by the four equations (3.37a,b) and (3.40a,b) as

$$q = -\Phi_{s1}(\theta_0), \quad r = \frac{\mathrm{d}\Phi_{s1}(\theta_0)}{\mathrm{d}\theta_0}\cos\psi, \quad \varphi_1(\theta_0) = \Phi_{s1}(\theta_0), \quad \tilde{\varphi}_1(\tilde{\theta}_0) = 0. \tag{3.42a-d}$$

Substituting (3.42a,b) into (3.18) for q and r, we obtain the solution p which is undisturbed initially (or $p(Y - c_y\tau) \to 0$ as $\tau \to -\infty$) as

$$p = \begin{cases} [\Phi_{s1}(\theta_0) - \Phi_{s1}(-\infty)]P & \text{for } 0 \le \psi < \cos^{-1}(1/v), \\ [\Phi_{s1}(\theta_0) - \Phi_{s1}(\infty)]P & \text{for } \cos^{-1}(1/v) < \psi \le \pi, \end{cases} \tag{3.43a}$$

where

$$P = \frac{v\left(c_y^2\dfrac{\mathrm{d}M}{\mathrm{d}v} - c_y\sqrt{v^2 + c_y^2 - 1}\dfrac{\mathrm{d}C}{\mathrm{d}v} - vM\right)}{(c_y^2 - c_0^2)\dfrac{\mathrm{d}E}{\mathrm{d}v}}, \tag{3.43b}$$

and

$$c_0 \equiv \pm\sqrt{\frac{vE}{\mathrm{d}E/\mathrm{d}v}}, \tag{3.43c}$$

with E being defined by (2.11d). Note that P diverges for $c_y = c_0$, in which case a different analysis with finite order of the phase shift p should be made.

Substituting (3.42c,d) into (3.31) and (3.35), we have the leading-order far-field solution as

$$\phi_{F1} = \Phi_{s1}(\theta), \tag{3.44a}$$

$$\eta_{F2} = \mathrm{H}_{s2}(\theta), \tag{3.44b}$$

where Φ_{s1} and H_{s2} are the first and second-order solutions of the small-amplitude solitary wave solution in ε given by (2.14a,d).

Next, the six unknowns $\varphi_2(\theta_0)$, $\widetilde{\varphi}_2(\widetilde{\theta}_0)$, $[\phi_{C2}-rx]_{x\to\pm\infty}$ and $[u_{C3}-x\partial u_{C3}/\partial x]_{x\to\pm\infty}$ are determined by the six equations (3.16), (3.21), (3.39a,b) and (3.41a,b). Solutions are

$$\varphi_2(\theta_0) = -P_s\frac{\mathrm{d}\Phi_{s1}(\theta_0)}{\mathrm{d}\theta_0}, \tag{3.45a}$$

$$\widetilde{\varphi}_2(\widetilde{\theta}_0) = -\frac{\widetilde{P}_s}{\kappa}\frac{\mathrm{d}\Phi_{s1}(\kappa\widetilde{\theta}_0)}{\mathrm{d}\widetilde{\theta}_0}, \tag{3.45b}$$

where

$$\kappa = \frac{\sin\psi}{\sin\widetilde{\psi}} = \frac{1-v\cos\psi}{1-v\cos\widetilde{\psi}}, \tag{3.46}$$

$$\begin{aligned} P_s = \frac{1}{2}\Bigg\{&\left[P\left(-\frac{\mathrm{d}C}{\mathrm{d}v}+\frac{c_y}{\sqrt{v^2+c_y^2-1}}\left(\frac{\mathrm{d}M}{\mathrm{d}v}-\frac{vM}{c_y^2}\right)\right)+C_B-\frac{c_y}{\sqrt{v^2+c_y^2-1}}\left(M_B-\frac{M}{c_y^2}\right)\right](1-v\cos\psi) \\ &+\left(C_U-\frac{c_y}{\sqrt{v^2+c_y^2-1}}M_U\right)\cos\psi\Bigg\}, \end{aligned} \tag{3.47a}$$

$$\begin{aligned} \widetilde{P}_s = \frac{1}{2}\Bigg\{&\left[P\left(-\frac{\mathrm{d}C}{\mathrm{d}v}-\frac{c_y}{\sqrt{v^2+c_y^2-1}}\left(\frac{\mathrm{d}M}{\mathrm{d}v}-\frac{vM}{c_y^2}\right)\right)+C_B+\frac{c_y}{\sqrt{v^2+c_y^2-1}}\left(M_B-\frac{M}{c_y^2}\right)\right](1-v\cos\psi) \\ &+\left(C_U+\frac{c_y}{\sqrt{v^2+c_y^2-1}}M_U\right)\cos\psi\Bigg\}. \end{aligned} \tag{3.47b}$$

Substituting (3.45) into (3.32) and (3.36), we have the next-order far-field solution as

$$\phi_{F2} = \begin{cases} 0 \quad \text{for } X>0, \\ -P_s\dfrac{\mathrm{d}\Phi_{s1}(\theta)}{\mathrm{d}\theta}-\dfrac{\widetilde{P}_s}{\kappa}\dfrac{\mathrm{d}\Phi_{s1}(\kappa\widetilde{\theta})}{\mathrm{d}\widetilde{\theta}} \quad \text{for } X<0, \end{cases} \tag{3.48a}$$

$$\eta_{F3} = \begin{cases} 0 \quad \text{for } X>0, \\ -P_s\dfrac{\mathrm{dH}_{s2}(\theta)}{\mathrm{d}\theta}-\widetilde{P}_s\dfrac{\mathrm{dH}_{s2}(\kappa\widetilde{\theta})}{\mathrm{d}\widetilde{\theta}} \quad \text{for } X<0. \end{cases} \tag{3.48b}$$

The solution obtained in this section is physically interpreted as follows. The core solution up to $O(\varepsilon^2)$ given by (3.12) and (3.14) represents modulation and associated phase shift of the extended solitary wave solution (文献[13]の補遺参照) whose parameters (v^*,B,U) are subject to small and slow variations. Deviations of the parameters (v^*,B,U) from their initial values $(v,1,0)$ are $(\varepsilon^2\partial p/\partial\tau, \varepsilon^2\partial q/\partial\tau, \varepsilon^2 r)$, and they are expressed in terms of the surface displacement $\varepsilon^2\mathrm{H}_{s2}(\theta_0)$ of the small-amplitude solitary wave at $X=0$, i.e.

$$\begin{pmatrix} v^* \\ B \\ U \end{pmatrix} = \begin{pmatrix} v \\ 1 \\ 0 \end{pmatrix} + \varepsilon^2\begin{pmatrix} (v\cos\psi-1)P \\ (1-v\cos\psi) \\ \cos\psi \end{pmatrix}\mathrm{H}_{s2}(\theta_0). \tag{3.49}$$

The associated phase shift, or the degree of translation is represented by εp, where p is given by (3.43). The total phase shift $\varepsilon p|_{\tau\to\infty}$ is

$$\varepsilon p\big|_{\tau\to\infty} = \begin{cases} \dfrac{2\sqrt{6}}{3}\varepsilon P & \text{for } 0 \le \psi < \cos^{-1}(1/v), \\ -\dfrac{2\sqrt{6}}{3}\varepsilon P & \text{for } \cos^{-1}(1/v) < \psi \le \pi. \end{cases} \tag{3.50}$$

Figure 2 shows the profile of $p|_{\tau\to\infty}$ as a function of ψ for $a = 0.3$ and 0.6. Note that $p|_{\tau\to\infty}$ diverges when $c_y = c_0$ as already mentioned after (3.43b).

The far-field solution is given by (3.44) and (3.48). The leading-order solution (3.44) is the small-amplitude solitary wave solution itself, while the next-order solution (3.48) represents two physical phenomena. The first terms on the right-hand sides of (3.48) for $X < 0$ represent translation of the small-amplitude solitary wave by a finite distance P_s (the phase shift is $\sqrt{3}\,\varepsilon P_s/4$) due to interaction with the finite-amplitude solitary wave. Figure 2 shows P_s as a function of ψ for $a = 0.3$ and 0.6. The second terms on the right-hand sides of (3.48) for $X < 0$ represent generation of the residual wave due to inelastic nature of the interaction. The residual wave has surface profile of $\mathrm{sech}^2 \tanh$ type, and propagates at an inclination angle $\tilde{\psi}$ to the x axis. Its amplitude $\tilde{P}_s$ is shown as a function of ψ for $a = 0.3$ and 0.6 in figure 2.

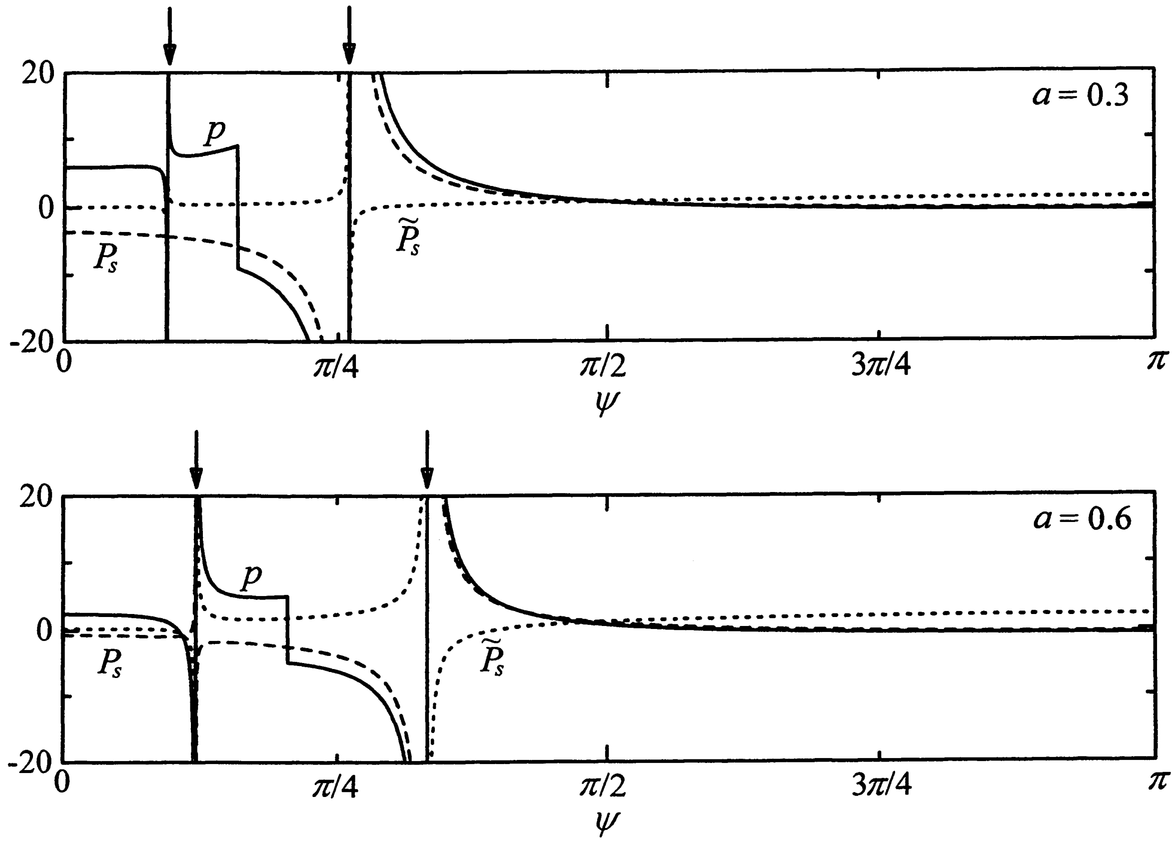

図2　有限振幅孤立波の位相のずれ $p|_{\tau\to\infty}$ (図中には単に p と記した), 小振幅孤立波の位相のずれ P_s, 放射波の振幅 $\tilde{P}_s$ を，相互作用前の２孤立波がなす角 ψ の関数としてプロットしたもの（上図：$a = 0.3$，下図：$a = 0.6$）．各図上部の矢印は，$c_y = c_0$ となる場合を表す．

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